Comparing Tent and Sawtooth 1-Dimensional Maps

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This paper will examine a 1-dimensional map we have created and named the Sawtooth Map. The Sawtooth Map will be compared to the behavior of the often examined Tent Map due to the structural similarity. Though the two maps are similar in shape, they exhibit highly differing behavior based on identical starting parameters and provide insight into stable, unstable, and chaotic behavior. This map appears to be an interesting case and we believe that it could be implemented as an approximation for more advanced 1-D maps.

Introduction: One-dimensional maps are deterministic functions whose values are determined by the function’s previous output. The generalized form is:

\[ p_{n+1} = f(P_n) \quad n = 0, 1, 2, ... \] (1)

Though these functions are often used to demonstrate shifts between stable, unstable, and chaotic behavior, some maps such as the logistic map can be used to make simple models of population growth and decline. This paper will examine two one-dimensional maps: the Tent Map and the Sawtooth Map. Equations for these functions are given below.

**Tent-Map**

\[ P_{n+1} = \begin{cases} 
  aP_n & \text{if } P_n \leq \frac{1}{2} \\
  a(1-P_n) & \text{if } P_n \geq \frac{1}{2} 
 \end{cases} \] (2)

**Sawtooth Map**

\[ P_{n+1} = \begin{cases} 
  2aP_n & \text{if } P_n \leq \frac{1}{2} \\
  2a(0.5 - P_n) & \text{if } \frac{1}{4} < P_n \leq \frac{1}{2} \\
  2aP_n - a & \text{if } \frac{1}{2} < P_n \leq \frac{3}{4} \\
  a(1-P_n) & \text{if } \frac{3}{4} < P_n 
 \end{cases} \] (3)

The tent map was chosen as an introductory example of a 1-D iterated map. The sawtooth map, created by the authors, was made to investigate the possibility of behavioral patterns between two maps with similar physical structure. The Sawtooth Map, shown in figures 2 and 5 indicates the physical form of the Sawtooth Map. The Tent Map has a similar physical form with only one “hump.”

These functions will be evaluated on the basis of their numerical behavior. First, numerical iteration will be used to discuss convergent, oscillatory, and chaotic behavior. Graphical iteration will be used to show how the values of the functions are deterministic. This paper will also consider the properties of the function such as the fixed points of each function, orbit diagrams, Lyapunov number, and sensitivity. The authors hope to show that the maps exhibit similar behavior under certain circumstances but are not analogous under all circumstances.

**Numerical Iteration**

Numerical iteration involves choosing initial \( P \) values and an \( a \) value for the function, and then using the function’s output to determine its input and reviewing the subsequent behavior. The behavior of the function under certain starting conditions and initial values is particularly descriptive.

Using numerical iteration, converging, oscillating, and even chaotic behavior can be understood. By plotting value versus the number of the iteration, the value of convergence or iteration at which chaos begins can be demonstrated graphically. Oscillating behavior can also be seen, though it is often difficult to demonstrate higher than a period-2 orbit with this method. Figures 3 and 1 demonstrate numerically iterated values for the Sawtooth and Tent Maps, respectively, for different values of \( a \). The figures show the switch from convergent to chaotic behavior for both maps. The behavior occurs as the \( a \) value changes from .45 to .55 in the Sawtooth Map in figure 3 and from .85 to 1.05 in the Tent Map in figure 1.

**Graphical Iteration**

Graphical iteration is an analysis method that uses a plot of a function and a starting
point to graphically determine the subsequent values of $P$. The orbit can be used to show convergent behavior that is independent of iteration number and can also more effectively display periodic orbits. This method of displaying data also assist in demonstrating chaotic behavior but is not effective at showing changes in behavior with alpha such as a switch from instability to convergence. Figure 2 shows graphical iteration for the Sawtooth Map.

![Graphical Iteration of Sawtooth Map](image)

FIG. 2: Graphical Iteration of Sawtooth Map. Starting at $p_n = .65$, one can find the orbit by moving up to hit the map, over to hit the unity line, up to hit the map, over to the unity line, etc. In this case, the orbit eventually hits the map at $p_n = 0.5$ at which point it goes over to the identity line at $(0, 0)$ which is a fixed point and there stops. This is an unusual behavior characteristic of a “2-humped” map with a 3rd 0 intersect.

As is visible in figure 2, the Sawtooth Map may appear to act like a Tent Map over the range of 0 to .5, but the intersection of the identity line with the plot of the function indicates that the behavior of the Tent and Sawtooth Maps differs significantly by both changing the number of fixed points and the orbit that is created for a given $a$ value and starting condition.

**Fixed Points**

A fixed point occurs when the input of a function maps to itself. These points can be stable, meaning that small disturbances in the input do not alter convergence. If the points are unstable, then a small disturbance in the input causes the subsequent values to move away from the fixed point.

Fixed points characterize a map by helping to show the quantitative behavior of the map. In the case of the tent map when it is used to approximate a logistic map, the non-zero fixed point approximates a population value that will neither increase nor decrease. Stable fixed points indicate that the orbits will generally converge to a single point. A map with unstable fixed points has largely divergent behavior that can very quickly become chaotic and hard to characterize.

The fixed points of the Sawtooth Map are stable when $a$ is below 0.5 or when the initial slope of the map is below the identity line. However, the fixed points of the Tent Map are stable when the $a$ value is below 1, because the Tent Map has half the slope of the Sawtooth Map, it intersects the identity line at a higher value. In this case, the Tent Map and the Sawtooth Map act similarly, and predictably based on the doubling of the number of peaks in the Sawtooth Map.

One interesting thing that distinguishes the Sawtooth Map from the Tent Map is the ability to hit a fixed point after one iteration. With an initial condition of $p_n = .65$, the orbit shoots to the identity line at $(0, 0)$, hitting the zero fixed point. This can also occur at the boundary condition of the Tent Map, i.e. $p_0 = 1$ but the boundary conditions are generally not studied in 1D maps. This behavior only occurs when the valley between the two sawtooth peaks maps to zero or another fixed point value.

When the $a$ is larger that 0.5 for the Sawtooth Map, the fixed points are highly unstable. These points can be found analytically or graphically at the intersection of the map and the identity line.

The stability of the points is determined by the absolute value of the slope at the fixed point. The slopes for the sawtooth map is either $2a$ or $-2a$. If $a > \frac{1}{2}$, the fixed points are all unstable.

It is for this reason that $a$ determines whether the map
becomes chaotic. At certain values of \(a\), the fixed points are unstable causing wild though deterministic behavior. We can see the direct influence of \(a\) by looking at the bifurcation diagram in figure 4.

A bifurcation diagram shows the behavior of the orbit for different values of \(a\). Figure 4 is the bifurcation diagram of the Sawtooth Map. Once \(a\) increases past a value of 0.5, we can see the emergence of periodic type functions. We can see an example of this pseudo-periodic behavior in figure 5. Additionally, it is interesting to note that at all \(a\) values less than 0.5 cause the iterated values to converge to 0.

**Periodic Orbits** As we can see in the bifurcation diagram in figure 4 and an example of graphical iteration in figure 5, there are periodic tendencies in the Sawtooth Map. To find the points of true periodic behavior, we must look for the Period-2 orbits. Period-\(n\) orbits are defined as points which map to themselves after \(n\) iterations. For a period-2 orbit this means that \(p = f(f(p))\).

These can be found analytically for both the Sawtooth and Tent Maps. The period-2 orbits for the Sawtooth Map are shown here.

\[
P = \begin{cases} 
0 & \text{if } P \leq \frac{1}{2} \\
\frac{a-2a^3}{1-4a^2} & \text{if } \frac{1}{2} < P \leq \frac{1}{4} \\
\frac{-2a-4a^2}{1-4a^2} & \text{if } \frac{1}{4} < P \leq \frac{3}{4} \\
\frac{2a}{1+2a} & \text{if } \frac{3}{4} < P
\end{cases}
\]

(4)

Period-2 orbits always come in pairs. If \(p = f(f(p))\) and \(f(p) = q\), then \(q = f(f(q))\) in a deterministic system. The graphical iteration of the map would show perfect squares at a period-2 orbit. The orbit would bounce from \(p\) to \(q\) to \(p\). Periodic orbits also have a component of stability. It is given by the product of the slopes at both points in the orbit. The orbit is unstable for values of \(a > 0.25\) but values of \(a\) less than 0.5 converge to zero so the period-2 orbits are always unstable.

**Lyapunov Number** The Lyapunov number can be used to determine when a function becomes chaotic. The number is a factor of the slopes of the map at all points in the orbit as shown in the following equation.

\[
L = \lim_{n \to \infty} (|f''(p_1)||f''(p_2)||…|f''(p_n)|)^{1/n}
\]

(5)

A Lyapunov value over 1 shows that the orbit which is not periodic is chaotic. In the case of the Sawtooth Map, the Lyapunov Number is equal to twice the value of \(a\). Figure 5 demonstrates a chaotic orbit. Because the slope of the Sawtooth Map is always 2\(a\) or \(-2a\), the Lyapunov number is always 2\(a\) according to equation 5.

**Sensitivity** Evaluating maps at two start values with a small difference and determining the number of iterations required before diverging allows us to determine the sensitivity of the parameters of a map. The Sawtooth Map has an unusual sensitivity characteristic shown in figure 6. There is a decrease in the number of iterations it takes for two orbits to diverge at a separation distance of \(10^{-6}\). This interesting behavior is perhaps due to the pseudo-oscillatory nature of most orbits.

**Conclusion** Although the Tent Map and Sawtooth
FIG. 6: Sensitivity Plot of Sawtooth Map. By plotting the number of iterations it takes for two initial points separated by $10^{-n}$ to separate by more than 0.5, we can see a characterization of the sensitivity of the map. The sawtooth map is very sensitive to small changes in the initial conditions. The unusual dip at the separation distance of $10^{-6}$ is an unusual phenomena characteristic of the Sawtooth Map.

Map appear to be similar in form, the maps have highly differing behavior. The Sawtooth Map is unstable except at the 0 fixed point for all inputs and $a$ values while the Tent Map shows stability at its fixed points for some $a$ values. The small differences in the Sawtooth Map produce behavior that is highly chaotic and very interesting to analyze.

The tent map and sawtooth map have a number of interesting similarities. First, the shapes of the two maps are very similar, with the sawtooth map containing two sharp peaks while the tent map only contains one. The fixed points of each both become unstable after $a$ becomes to large (greater than .5 for the sawtooth map and 1 for the tent map). The unstable $a$ value seems to come from the relationship between slope of the identity line and the absolute slope value of the function itself. Second, they both exhibit period 2 orbits. However, these orbits are overshadowed by convergence very quickly when the value of $a$ becomes low.

Despite their similarities, the two maps exhibit a number of striking differences. Most notably, the sawtooth map is highly unstable over a much larger range than the tent map. This leads to seemingly chaotic orbits for many start conditions and $a$ values. Second, the Lyapunov number for each function differs by a factor of 2. As with the $a$ value, the Lyapunov number difference occurs because the slope of the sawtooth map is double that of the tent map for a given $a$ value.

In the future, we believe it would be interesting to extend this analysis to maps with 3 or more peaks. We would hope to determine whether or not the behavior of the map changes consistently as the number of peaks is increased. Our initial hypothesis states that there will be more interesting periodic orbits as the number of peaks increases. We also believe that an “N-shaped” map with a final peak that terminates at a non-zero value could have interesting behaviors. Additionally, as the tent map can be used to approximate the logistic map, we believe the sawtooth map could be used to approximate quadratic maps to some degree of accuracy. The interesting sensitivity plot shown in figure ?? is another area that requires further analysis.

In general, the authors have found the Sawtooth Map to be an interesting area of study. It’s simplified physical form and interesting behavior make it a worthwhile area of research.